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THE INTERSECTION OF MATROIDS AND ANTIMATROIDS*

Bernhard KORTE
Abteilung Operations Research, Universität Bonn, Nassestrasse 2, D-5300 Bonn 1, F.R. Germany

László LOVÁSZ
Budapest

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Antimatroids are combinatorial structures abstracting some properties of convexity, and in a sense dual to matroids. Greedoids are common generalizations of matroids and antimatroids. We introduce a general operation to produce a greedoid from a matroid and an antimatroid on the same ground set. Greedoids arising by this operation are called trimmed matroids. Many known classes of greedoids are shown to be trimmed matroids. We derive two submodularity properties of trimmed matroids and a subclass of them called polymatroid greedoids. These are used to verify the properties of a rather elaborate counterexample, which shows that certain local properties do not characterize trimmed matroids and polymatroids greedoids (as was conjectured in an earlier paper).

1. Introduction

Greedoids were introduced by the authors as common relaxations of matroids, antimatroids and other combinatorial structures with exchange properties (for more on their motivation, see Korte and Lovász [7, 8]).

In this paper we define an operation which produces a greedoid from any matroid-antimatroid pair on the same underlying set. We prove that the greedoids arising by this construction are exactly those which are formed by the common feasible sets of a matroid and an antimatroid. (One should point out that the common feasible sets of a matroid and an antimatroid do not form a greedoid in general.) We also show that these greedoids coincide with the "trimmed matroids" introduced in Korte and Lovász [10].

Many of the most important examples of greedoids belong to the class of so-called interval greedoids and in fact most to the class of trimmed matroids. The operation introduced in this paper gives very natural representations for many of these classes.

In Korte and Lovász [11] we gave an inclusion chart of many subclasses of interval greedoids and showed that all inclusions are proper and that all but at

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most one of the inclusion relations between these classes were shown in the chart. This was done by exhibiting a "typical member" for each class, which only belongs to this class and its superclasses. At the root of this inclusion chart we had two subclasses of greedoids, namely local poset greedoids and trimmed matroids. Therefore it was of certain interest whether there was a relationship between these two classes. We were able to show by a simple example that not all trimmed matroids are local posets. We are now able to answer the reverse question in the negative by a rather elaborate example which is a local poset greedoid but not a trimmed matroid.

In Korte and Lovász [10] we associated a greedoid with every polymatroid and showed that every polymatroid greedoid has three local properties, which we called local intersection, local union and local augmentation. Many important properties of polymatroid greedoids followed already from these three, and therefore we conjectured that polymatroid greedoids can be characterized by these properties. This was believed so much the more since local poset greedoids as a proper superclass could be characterized by the first and the second property. In this paper we disprove the conjecture by the same counterexample as mentioned above.

In these proofs we make use of two results showing that the rank functions of polymatroid greedoids and trimmed matroids enjoy stronger submodularity than the rank functions of greedoids in general. These facts may be of independent interest.

2. Definitions and basic facts about greedoids

A set system $(E, \mathcal{F})$ on a finite ground set E with $\mathcal{F} \subseteq 2^E$ is called an *accessible set system* if the following hold:

(H1) $\emptyset \in \mathcal{F}$

(H2) for all $X \in \mathcal{F}$, $X \neq \emptyset$, there exists an $x \in X$ such that $X - x \in \mathcal{F}$.

It is a *greedoid* if in addition (H3) holds:

(H3) if $X, Y \in \mathcal{F}$ and $|X| = |Y| + 1$, then there exists an $x \in X - Y$ such that $Y \cup x \in \mathcal{F}$.

A strengthening of this definition leads to the definition of *matroids*, namely if one replaces the accessibility axiom (H2) by the subclusivity axiom

(H2′) if $Y \subseteq X \in \mathcal{F}$ then $Y \in \mathcal{F}$.

An accessible set system is called an *antimatroid* if it is closed under union. It is easy to see that antimatroids are special greedoids. Antimatroids were introduced by Edelman [3] and Jamison [6], and also studied as special greedoids in Korte and Lovász [9].

An important property which holds for many greedoids is the *interval property*:

> if $A, B, C \in \mathcal{F}$ with $A \subseteq B \subseteq C$ and $x \in E - C$ such that $A \cup x \in \mathcal{F}$, $C \cup x \in \mathcal{F}$ then it follows that $B \cup x \in \mathcal{F}$.

Actually, all greedoids discussed in this paper enjoy this property. Greedoids with the interval property are called *interval greedoids*. An equivalent property defining interval greedoids is that if $X, Y \subseteq Z$ and $X, Y, Z \in \mathcal{F}$ then $X \cup Y \in \mathcal{F}$.

The set of all possible extensions of a set $A \in \mathcal{F}$ is denoted by

$$\Gamma(A) := \{x \in E - A : A \cup x \in \mathcal{F}\}.$$

For a greedoid $(E, \mathcal{F})$ we define its *k-truncation* $(E, \mathcal{F}^{(k)})$ by

$$\mathcal{F}^{(k)} := \{X \in \mathcal{F} : |X| \leq k\}$$

The *contraction of* a set $U \in \mathcal{F}$ results again in a greedoid $(E - U, \mathcal{F}/U)$ where

$$\mathcal{F}/U := \{X \subseteq E - U : X \cup U \in \mathcal{F}\}.$$

For greedoids we can define the (*independence*) *rank* of a set $X \subseteq E$ as

$$r(X) := \max\{|A| : A \subseteq X, A \in \mathcal{F}\}.$$

This non-negative function has the following properties for $X, Y \subseteq E$ and $x, y \in E$:

(R0) $r(\emptyset) = 0$

(R1) $r(X) \leq |X|$

(R2) $X \subseteq Y$ then $r(X) \leq r(Y)$

(R3) if $r(X) = r(X \cup x) = r(X \cup y)$ then $r(X) = r(X \cup x \cup y)$.

Conversely, a function $r : 2^E \to \mathbb{Z}$ satisfying (R0), (R1), (R2) and (R3) defines a unique greedoid. Again, these axioms are direct relaxations of the rank definition of matroids, for which in addition we have the *unit increase property*

(R4) $r(X \cup x) \leq r(X) + 1$ for $X \subseteq E$, $x \in E$.

Properties (R3) and (R4) together are equivalent to the submodularity property

(R4′) $r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y)$.

Hence, in general the greedoid rank function is not submodular.

Another way of extending the definition of the rank function of matroids to greedoids is to introduce the (*basis*) *rank* of a set $X \subseteq E$ as

$$\beta(X) := \max\{|X \cap B| : B \in \mathcal{F}\}.$$

Clearly, $\beta(X) \geq r(X)$. A set is called *rank feasible* if $\beta(X) = r(X)$. We denote the family of all rank feasible sets by $\mathcal{R}$. Clearly $\mathcal{F} \subseteq \mathcal{R}$. As an easy consequence of (H2′) we have $\mathcal{R} = 2^E$ iff $(E, \mathcal{F})$ is a matroid.

As for matroids, we can define a suitable *closure operator* for a greedoid $(E, \mathcal{F})$ by

$$\sigma_{\mathcal{F}}(X) = \sigma(X) := \{x \in E : r(X \cup x) = r(X)\}.$$

A set is called *closed* if $X = \sigma(X)$. We call a set X *closure feasible* if $X \subseteq \sigma(A)$ implies $X \subseteq \sigma(B)$ for $A \subseteq B \subseteq E$. The family of all closure feasible sets will be denoted by $\mathcal{C}$. We have $\mathcal{C} \subseteq \mathcal{R}$ and $\mathcal{C} = \mathcal{R}$ iff the greedoid has the interval property.

We can also define greedoids as languages. A *language* $\mathscr{L}$ over a finite ground set E (which is called an *alphabet*, is a collection of *finite sequences* of elements of E. The elements of E are called *letters*. We call the sequences *strings* or *words* and we often denote them by small greek letters. The notation $(E, \mathscr{L})$ will be used for languages. A language is called *simple* if no letter is repeated in a word. For simple languages we denote by $|\alpha|$ the length of the word α and by $\tilde{\alpha}$ its underlying set. A simple language $(E, \mathscr{L})$ is called a greedoid if the following hold:

(G1) $\emptyset \in \mathscr{L}$

(G2) if $x_1 \ldots x_k \in \mathscr{L}$ then $x_1 \ldots x_i \in \mathscr{L}$ for $1 \leq i \leq k$

(G3) if $x_1 \ldots x_k \in \mathscr{L}$ and $y_1 \ldots y_j \in \mathscr{L}$ with $k > j$ then there exists an x_i such that $y_1 \ldots y_j x_i \in \mathscr{L}$.

Obviously, the underlying set system of a language with properties (G1), (G2) and G(3) satisfies (H1), (H2) and (H3). It is not too difficult to see (cf. Korte and Lovász [7]) that a set system $(E, \mathscr{F})$ satisfying (H1), (H2) and (H3) induces a unique language $\mathscr{L}_{\mathscr{F}}$ with properties (G1), (G2), (G3). Thus, we can consider greedoids either as set systems or as languages. We will use it in the following concurrently.

Polymatroids are straightforward generalizations of matroids. Let E be a finite ground set and $f: 2^E \to \mathbb{Z}_+$. We call (E, f) a *polymatroid* if (R0), (R1), and (R4') hold. Every polymatroid (E, f) gives rise to an associated greedoid $(E, \mathscr{L})$ with

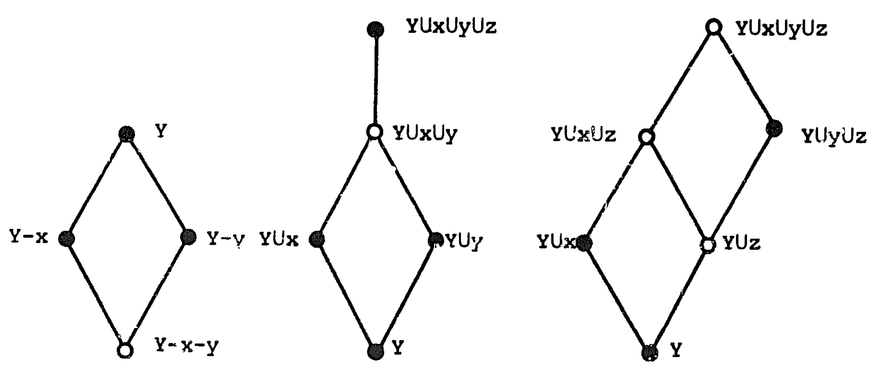
$$\mathscr{L} := \{x_1 \ldots x_k : f(x_1, \ldots, x_i) = i \quad \text{for all } 1 \leq i \leq k\}.$$

$(E, \mathscr{L})$ is called a *polymatroid greedoid.*

In Korte and Lovász [10] we have introduced three properties (A), (B) and (C) and proved that they hold for polymatroid greedoids.

(A) *Local intersection property*:
If $Y \subseteq E$, $x, y \in Y$ and $Y, Y-x, Y-y \in \mathscr{F}$, then $Y-x-y \in \mathscr{F}$.

(B) *Local union property*:
If $Y \subseteq E, x, y, z \in E-Y$, and $Y, U \cup x$, $Y \cup y$, $Y \cup x \cup y \cup z \in \mathscr{F}$, then $Y \cup x \cup y \in \mathscr{F}$.



Property A Property B Property C

Fig. 1.

(C) *Local augmentation property*:
If $Y \subset E$, $x, y, z \in E - Y$, and Y, $Y \cup x$, $Y \cup y \cup z \in \mathcal{F}$, then one of $Y \cup z$, $Y \cup x \cup z$, and $Y \cup x \cup y \cup z$ is an element of $\mathcal{F}$.

Fig. 1 illustrates these properties by showing the relevant part of the boolean algebra 2^E. If the full points are elements of $\mathcal{F}$ then one of the light points must also be an element of $\mathcal{F}$.

These properties give rise to another definition: A greedoid $(E, \mathcal{F})$ is called a *local poset greedoid* if it has properties (A) and (B). For further details about local poset greedoids we refer again to Korte and Lovász [10, 11].

3. The meet of a matroid and an antimatroid

Let $(E, \mathcal{M})$ be a matroid and $(E, \mathcal{A})$ an antimatroid on the same underlying set. For each word $\alpha = x_1 \cdots x_k \in \mathcal{L}_{\mathcal{A}}$, we define a subword $\hat{\alpha}$ as follows. Let

$$\begin{aligned} i_1 &= \min\{i : \{x_i\} \in \mathcal{M}\}, \\ i_2 &= \min\{i : i > i_1, \quad \{x_{i_1}, x_i\} \in \mathcal{M}\}. \\ &\vdots \\ i_r &= \min\{i : i > i_{r-1}, \quad \{x_{i_1}, \ldots, x_i\} \in \mathcal{M}\}. \end{aligned}$$

and let $\hat{\alpha} = x_{i_1} x_{i_2} \ldots x_{i_r}$. So $\hat{\alpha}$ is the lexicographically smallest $\mathcal{M}$-basis of $\tilde{\alpha}$. We set $\hat{\mathcal{L}} = \{\hat{\alpha} : \alpha \in \mathcal{L}_{\mathcal{A}}\}$. The following lemma is straightforward.

Lemma 3.1. *If* $\alpha = x_1 \ldots x_k$ *then*

$$\hat{\alpha} = (x_i : x_i \notin \sigma_{\mathcal{M}}(x_1 \ldots x_{i-1})).$$

Moreover, $\sigma_{\mathcal{M}}(\hat{\alpha}) = \sigma_{\mathcal{M}}(\alpha)$ *and* $|\hat{\alpha}| = r_{\mathcal{M}}(\alpha)$.

Lemma 3.2. $(E, \hat{\mathcal{L}})$ *is an interval greedoid.*

Proof. Let $\hat{\alpha}, \hat{\beta} \in \hat{\mathcal{L}}$, $|\hat{\alpha}| < |\hat{\beta}|$. Let β_1 be the subword of β consisting of the letters in $\tilde{\beta} - \tilde{\alpha}$. Then $\alpha\beta_1 \in \mathcal{L}_{\mathcal{A}}$ by elementary properties of antimatroids. Obviously, $\widehat{\alpha\beta_1} = \hat{\alpha}\beta_2$, where β_2 is a subword of β_1.

Claim 1. $|\beta_2| \geq |\hat{\beta}| - |\hat{\alpha}|$. For by Lemma 3.1,

$$|\hat{\alpha}\beta_2| = r_{\mathcal{M}}(\alpha\beta_1) = r(\tilde{\alpha} \cup \tilde{\beta}) \geq r(\beta) = |\hat{\beta}|.$$

Claim 2. β_2 is a subword of $\hat{\beta}$. Suppose not, and let x be the first letter of β_2 not in $\hat{\beta}$. Write $\beta = \beta_3 x \beta_4$, $\beta_1 = \beta_5 x \beta_6$. By definition, $\tilde{\beta}_5 = \tilde{\beta}_3 - \tilde{\alpha}$. By Lemma 3.1, $x \in \sigma_{\mathcal{M}}(\beta_3)$. Hence $x \in \sigma_{\mathcal{M}}(\tilde{\alpha} \cup \tilde{\beta}_3) = \sigma_{\mathcal{M}}(\tilde{\alpha} \cup \tilde{\beta}_5)$, and hence $x \notin \widehat{\alpha\beta_1} = \hat{\alpha}\beta_2$, a contradiction.

By a result of Björner [1], Claims 1 and 2 imply that $(E, \mathscr{L})$ is an interval greedoid.

We call this interval greedoid $(E, \mathscr{L})$ the *meet* of $\mathscr{M}$ and $\mathscr{A}$, and denote it in the unordered form by $(E, \mathscr{M} \wedge \mathscr{A})$.

The following lemma is essentially a rewriting of the definition of $\mathscr{M} \wedge \mathscr{A}$ in terms of the feasible sets.

Lemma 3.3. *Let $(E, \mathscr{M})$ be a matroid $(E, \mathscr{A})$ an antimatroid, $X \in \mathscr{M} \wedge \mathscr{A}$ and $x \in E - X$. Then $X \cup x \in \mathscr{M} \wedge \mathscr{A}$ iff $x \notin \sigma_{\mathscr{M}}(X)$ and there exists a set $U \in \mathscr{A}$ such that $x \in U$ and $U - x \subseteq \sigma_{\mathscr{M}}(X)$.*

Notice that, for any $x \notin \sigma_{\mathscr{M}}(X)$, a set $U \in \mathscr{A}$ with $x \in U$ and $U - x \in \sigma_{\mathscr{M}}(X)$ exists iff $x \notin \sigma_{\mathscr{A}}(\sigma_{\mathscr{M}}(X))$. Hence we can formulate this lemma as follows:

Corollary 3.4. *Let $(E, \mathscr{M})$ be a matroid and $(E, \mathscr{A})$, an antimatroid. Then for each $X \in \mathscr{M} \wedge \mathscr{A}$,*

$$\sigma_{\mathscr{M} \wedge \mathscr{A}}(X) = \sigma_{\mathscr{A}}(\sigma_{\mathscr{M}}(X)).$$

Remark. The previous corollary yields a polynomial time algorithm to check whether a word $x_1 \ldots x_k$ is feasible in $\mathscr{M} \wedge \mathscr{A}$, provided $(E, \mathscr{M})$ and $(E, \mathscr{A})$ are given by feasibility oracles. In fact, $x_1 \ldots x_k \in \mathscr{L}_{\mathscr{M} \wedge \mathscr{A}}$ if and only if for all $1 \leqslant i \leqslant k$,

$$x_i \notin \sigma_{\mathscr{A}}(\sigma_{\mathscr{M}}(x_1 \ldots x_{i-1})).$$

(The closures $\sigma_{\mathscr{A}}$ and $\sigma_{\mathscr{M}}$ are easily computed from the feasibility oracles.)

It turns out that greedoids arising in this form are just the same as *trimmed matroids* as defined in Korte and Lovász [10]. The definition given there is the following.

Let $(\mathscr{E}, \mathscr{M})$ be a matroid and let for each $e \in E$, $\mathscr{T}(e) \subseteq 2^E$. Then the *trimming* of $(E, \mathscr{M})$ by $\mathscr{T}$ is defined as the language

$$\mathscr{M}\langle \mathscr{T} \rangle = \{e_1 \ldots e_k\colon \quad \text{for all } 1 \leqslant j \leqslant k, \quad \exists T \in \mathscr{T}(e_j) \quad \text{with} \quad T \subseteq \sigma_{\mathscr{M}}(e_1 \ldots e_{j-1})\}.$$

The following theorem asserts the equivalence of these two definitions and also characterizes trimmed matroids as intersections of matroids and antimatroids.

Theorem 3.5. *For any greedoid $(E, \mathscr{F})$ the following are equivalent:*

(i) *there exist a matroid $(E, \mathscr{M})$ and an antimatroid $(E, \mathscr{A})$ such that $\mathscr{F} = \mathscr{M} \wedge \mathscr{A}$.*

(ii) *there exist a matroid $(E, \mathscr{M})$ and an antimatroid $(E, \mathscr{A})$ such that $\mathscr{F} = \mathscr{M} \cap \mathscr{A}$.*

(iii) *$(E, \mathscr{F})$ is a trimmed matroid.*

Remark. While the constructions involved in (i) and (iii) always yield greedoids, the common feasible sets of a matroid and an antimatroid do not form a greedoid in general. For example, $\mathcal{M} = \{1, 2, 3, \{1, 2\}, \{1, 3\}\}$ is a matroid on $E = \{1, 2, 3\}$, and $\mathcal{A} = \{2, 3, \{2, 3\}, \{1, 2\}, \{1, 2, 3\}\}$ is an antimatroid on this same set. Then $\mathcal{M} \cap \mathcal{A} = \{2, 3, \{1, 2\}\}$ is not a greedoid. Note that $\mathcal{M} \wedge \mathcal{A} = \{2, 3, \{1, 2\}, \{1, 3\}\}$ is a greedoid.

Note that for any matroid $(E, \mathcal{M})$ and antimatroid $(E, \mathcal{A})$, we have

$$\mathcal{M} \cap \mathcal{A} \subseteq \mathcal{M} \wedge \mathcal{A} \subseteq \mathcal{M}.$$

To prove Theorem 3.5, we need the following lemma.

Lemma 3.6. *Let $(E, \mathcal{M})$ be a matroid, $(E, \mathcal{F})$ a greedoid, and $\mathcal{A} = \{\bigcup \mathcal{F}' : \mathcal{F}' \subseteq \mathcal{F}\}$. Suppose that $\mathcal{F} = \mathcal{M} \cap \mathcal{A}$. Then $\mathcal{F} = \mathcal{M} \wedge \mathcal{A}$.*

Proof. As noted, $\mathcal{F} \subseteq \mathcal{M} \wedge \mathcal{A}$. Conversely, let $X \in \mathcal{M} \wedge \mathcal{A}$. Choose an element $x \in X$ such that $X - x \in \mathcal{M} \wedge \mathcal{A}$. By induction we may assume that $X - x \in \mathcal{F}$. It follows from the definition of $\mathcal{M} \wedge \mathcal{A}$ that there exists a set $U \in \mathcal{F}$ such that $x \in U$ and $U - x \subseteq \sigma_{\mathcal{M}}(X - x)$. Choosing U as small as possible, we may assume that $U - x \in \mathcal{F}$. Augment $U - x$ to an $\mathcal{F}$-basis Y of $\sigma_{\mathcal{M}}(X - x)$. Then $Y \cup \{x\} = Y \cup U \in \mathcal{A}$ by the definition of $\mathcal{A}$. Moreover, $Y \in \mathcal{M}$ since $Y \in \mathcal{F}$. Since $x \notin \sigma_{\mathcal{M}}(Y) = \sigma_{\mathcal{M}}(X - x)$ we have $Y \cup \{x\} \in \mathcal{M}$. So $Y \cup \{x\} \in \mathcal{M} \cap \mathcal{A} = \mathcal{F}$. Now augument $X - x$ from $Y \cup \{x\}$ in the greedoid $\mathcal{F}$. Since $Y \subseteq \sigma_{\mathcal{M}}(X - x)$, $X - x \cup \{y\} \notin \mathcal{M}$ for $y \in Y - (X - x)$ and so $X - x \cup \{y\} \notin \mathcal{F}$. Thus we must have $X - x \cup \{x\} = X \in \mathcal{F}$. □

Proof of Theorem 3.5.
(ii)$\Rightarrow$(i): Assume that $\mathcal{F} = \mathcal{M} \cap \mathcal{A}_1$ for some matroid $(E, \mathcal{M})$ and antimatroid $(E, \mathcal{A}_1)$. Define $\mathcal{A} = \{\bigcup \mathcal{F}' : \mathcal{F}' \subseteq \mathcal{F}\}$. Observe that $\mathcal{F} \subseteq \mathcal{A}_1$ and hence $\mathcal{A} \subseteq \mathcal{A}_1$. So

$$\mathcal{F} = \mathcal{M} \cap \mathcal{A}_1 \supseteq \mathcal{M} \cap \mathcal{A}.$$

But also trivially

$$\mathcal{F} \subseteq \mathcal{M} \cap \mathcal{A}$$

and hence

$$\mathcal{F} = \mathcal{M} \cap \mathcal{A}.$$

So the assertion follows from Lemma 3.6.
(i)$\Rightarrow$(iii) Define, for each $e \in E$,

$$\mathcal{T}(e) = \{X \in \mathcal{A} : X \cup e \in \mathcal{A}\}.$$

Then $\mathcal{M}\langle \mathcal{T} \rangle = \mathcal{M} \wedge \mathcal{A}$. For, assume that $x_1 \cdots x_k \in \mathcal{L}_{\mathcal{M}}\langle \mathcal{T} \rangle$. Then for each j, there exists a set $T_j \in \mathcal{T}(x_j)$ such that $T_j \subseteq \sigma_{\mathcal{M}}(x_i \cdots x_{j-1})$. By the definition of

$\mathcal{T}(x_j)$, $T_j \in \mathcal{A}$ and $T_j \cup \{x_j\} \in \mathcal{A}$. Let t_1 be an $\mathcal{A}$-feasible ordering of T_1, $t_1 x t_2$ an $\mathcal{A}$-feasible ordering of $T_1 \cup \{x\} \cup T_2 \in \mathcal{A}$, etc. So we obtain a word

$$\alpha = t_1 x_1 t_2 x_2 \cdots t_k x_k \in \mathcal{L}_{\mathcal{A}}$$

where $\tilde{t}_j \subseteq T_j \subseteq \sigma_{\mathcal{M}}(x_1 \cdots x_{j-1})$. Hence $\hat{\alpha} = x_1 \cdots x_k$ and so $x_1 \cdots x_k \in \mathcal{L}_{\mathcal{M} \wedge \mathcal{A}}$.

Conversely, let $x_1 .. x_k \in \mathcal{L}_{\mathcal{M} \wedge \mathcal{A}}$. Then $x_1 \ldots x_k = \hat{\alpha}$ for some $\alpha = \alpha_1 x_1 .. \alpha_k x_k \in \mathcal{L}_{\mathcal{A}}$. Let $\gamma = \alpha_1 x_1 \ldots \alpha_j$ and set $T_j = \tilde{\gamma}$. Then $T_j \in \mathcal{T}(e_j)$ by the definition of $\mathcal{T}(e_j)$ and $T_j \subseteq \sigma_{\mathcal{M}}(x_1 \ldots x_{j-1})$ by the definition of $\hat{\alpha}$. Hence $x_1 \ldots x_k \in \mathcal{M}\langle \mathcal{T} \rangle$.

(iii)$\Rightarrow$(ii) This implication was informally mentioned in Korte and Lovász [10], but for sake of completeness we give the argument. Let $\mathcal{F} = \mathcal{M}\langle \mathcal{T} \rangle$. Define an antimatroid $\mathcal{A} = \{ \cup \mathcal{F}' : \mathcal{F}' \subseteq \mathcal{F}\}$. We claim that $\mathcal{M} \cap \mathcal{A} = \mathcal{F}$. Trivially, $\mathcal{M} \cap \mathcal{A} \supseteq \mathcal{F}$.

Conversely, let $X \in \mathcal{A} \cap \mathcal{M}$. Choose an element $x \in X$ such that $X - x \in \mathcal{A}$. Then trivially $X - x \in \mathcal{A} \cap \mathcal{M}$ and hence we may assume by induction that $X - x \in \mathcal{F}$. Also from $X \in \mathcal{A}$ it follows that there exists a set $U \in \mathcal{F}$ such that $U \subseteq X$ and $x \in U$. We may assume that $U - x \in \mathcal{F}$. By the definition of $\mathcal{M}\langle \mathcal{T} \rangle$, there exists a $T \in \mathcal{T}(x)$ such that $T \subseteq \sigma_{\mathcal{M}}(U - x) \subseteq \sigma_{\mathcal{M}}(X - x)$. So $X = X - x \cup \{x\} \in \mathcal{F}$ as claimed. $\square$

Remark. The operation $\mathcal{M} \cap \mathcal{A}$ yields interesting representations for some subclasses of greedoids.

(1) If $(E, \mathcal{M})$ is any matroid and $(E, \mathcal{A})$ is a poset greedoid then $(E, \mathcal{M} \wedge \mathcal{A})$ is a minimal F-geometry and conversely, every minimal F-geometry arises this way. F-geometries were introduced by Faigle [4, 5]. Minimal F-geometries and the relations between F-geometries and greedoids were studied in Korte and Lovász [11, 12].

(2) If G is a directed graph with root r, $(E, \mathcal{A})$ is the line search greedoid from this root (see Korte and Lovász [8]), and $(E, \mathcal{M})$ is the partition matroid defined by the partition of the edge-set into "in-stars", then $(E, \mathcal{M} \wedge \mathcal{A})$ is the directed greedoid of G. Instead of the partition matroid, we could take the graphic matroid of G, and get the same greedoid.

(3) If (E, f) is a polymatroid, $(E, \mathcal{M})$ the induced matroid, and $\mathcal{A} = \{x_1 \ldots x_k : f(x_1 \ldots x_{i+1}) \leq f(x_1 \ldots x_i) + 1 \text{ for all } 0 \leq i \leq k-1\}$, then $(E, \mathcal{M} \wedge \mathcal{A})$ is the polymatroid greedoid determined by (E, f).

(4) Consider any antimatroid $(E, \mathcal{A})$ and a partition matroid $(E, \mathcal{M})$ defined by the partition $\{V_1, \ldots, V_k\}$. Then $(E, \mathcal{M} \wedge \mathcal{A})$ is an interval greedoid, in which every basis meets every partition class V_i in exactly one element. Such a partition was called a *balanced partition* in Björner, Korte and Lovász [2]. Conversely, every interval greedoid with a balanced partition $E = V_1 \cup \cdots \cup V_r$ arises this way. For, let $(E, \mathcal{M})$ be the partition matroid defined by $\{V_1, \ldots, V_r\}$ and let $\mathcal{A} = \{ \cup \mathcal{F}' : \mathcal{F}' \subseteq \mathcal{F}\}$. We claim that $\mathcal{F} = \mathcal{A} \wedge \mathcal{M} = \mathcal{A} \cap \mathcal{M}$.

By Lemma 3.6, it suffices to prove that $\mathcal{F} = \mathcal{A} \cap \mathcal{M}$. Obviously, $\mathcal{F} \subseteq \mathcal{A} \cap \mathcal{M}$.

Let $X \in \mathscr{A} \cap \mathscr{M}$, and assume that $X \notin \mathscr{F}$. Since $X \in \mathscr{A}$, we can write $X = X_1 \cup X_2$, where $X_1 \in \mathscr{A}$, $X_2 \in \mathscr{F}$ and $|X_1| < |X|$. Using induction, we may assume that $X_1 \in \mathscr{F}$. Let B_1 be any basis of $(E, \mathscr{F})$ containing X_1, and let B_2 be any basis of $(E, \mathscr{F})$ obtained by augmenting X_2 from B_1.

It is easy to see, using the assumption that $\{V_1, \ldots, V_r\}$ is a balanced partition, that $B_2 = X_2 \cup (B_1 - \sigma_{\mathscr{M}}(X_2))$. Since $X_1 \cup X_2 \in \mathscr{M}$, this implies that $X_1 \subseteq B_2$. So X_1 and X_2 are feasible subsets of the basis B_2. Since $(E, \mathscr{F})$ is an interval greedoid, it follows that $X_1 \cup X_2 \in \mathscr{F}$.

4. Submodularity in trimmed matroids

In this section we prove that the submodularity property of the rank function of greedoids can be slightly strengthened for trimmed matroids and even further for polymatroid greedoids.

Lemma 4.1. *Let $(E, \mathscr{M})$ be a matroid, $(E, \mathscr{A})$ an antimatroid, and $(E, \mathscr{F})$ the associated trimmed matroid, i.e. $\mathscr{F} = \mathscr{M} \wedge \mathscr{A}$. Let $r_{\mathscr{M}}$ and $r_{\mathscr{F}}$ be the rank function of the matroid and the trimmed matroid respectively. Then for each $X \in \mathscr{A}$ we have*

$$r_{\mathscr{F}}(X) = r_{\mathscr{M}}(X).$$

Proof. Obviously, $r_{\mathscr{F}}(X) \leqslant r_{\mathscr{M}}(X)$. On the other hand if $X = \{x_1, \ldots, x_k\} \in \mathscr{A}$ then let $\alpha = x_1 \ldots x_k \in \mathscr{L}_{\mathscr{A}}$. Then $|\hat{\alpha}| = r_{\mathscr{M}}(X)$ by construction, and hence $r_{\mathscr{F}}(X) \geqslant |\hat{\alpha}| = r_{\mathscr{M}}(X)$. □

Lemma 4.2. *Let (E, f) be a polymatroid and let $(E, \mathscr{F})$ be the associated polymatroid greedoid. Then for every $X \subseteq E$,*

$$\beta(X) \leqslant f(X).$$

If equality holds, then $X \in \mathscr{R} = \mathscr{A}$.

Proof. Let B be a basis of $(E, \mathscr{F})$ such that $\beta(X) = |X \cap B|$. Then by Lemma 4.2 of Korte and Lovász [10]

$$f(X \cap B) \geqslant |X \cap B|,$$

and hence

$$f(X) \geqslant f(X \cap B) \geqslant |X \cap B| = \beta(X).$$

If equality holds, then $f(X \cap B) = |X \cap B|$ and hence by Lemma 4.2 of Korte and Lovász [10] again, $X \cap B \in \mathscr{F}$, so

$$r(X) \geqslant |X \cap B| = \beta(X)$$

and so $X \in \mathscr{R} = \mathscr{A}$. □

Theorem 4.3. *If $(E, \mathcal{F})$ is a trimmed matroid and $S_1, S_2 \in \mathcal{A}$ then*

$$\beta(S_1 \cap S_2) + r(S_1 \cup S_2) \leqslant r(S_1) + r(S_2).$$

Proof. Let $\mathcal{F} = \mathcal{M} \wedge \mathcal{A}$. Let B be a basis of $(E, \mathcal{F})$ such that $|S_1 \cap S_2 \cap B| = \beta(S_1 \cap S_2)$. Then $B \in \mathcal{M}$. So $\beta(S_1 \cap S_2) = |S_1 \cap S_2 \cap B| \leqslant r_{\mathcal{M}}(S_1 \cap S_2)$.

Moreover

$$r(S_i) = r_{\mathcal{M}}(S_i) \quad \text{and} \quad r(S_1 \cup S_2) = r_{\mathcal{M}}(S_1 \cup S_2)$$

by Lemma 4.1, and hence

$$\beta(S_1 \cap S_2) + r(S_1 \cup S_2) \leqslant r_{\mathcal{M}}(S_1 \cap S_2) + r_{\mathcal{M}}(S_1 \cup S_2)$$
$$\leqslant r_{\mathcal{M}}(S_1) + r_{\mathcal{M}}(S_2) = r(S_1) + r(S_2). \quad \square$$

Theorem 4.4. *Let $(E, \mathcal{F})$ be a polymatroid greedoid, and $S_1, S_2 \in \mathcal{A}$ such that*

$$\beta(S_1 \cap S_2) + r(S_1 \cup S_2) = r(S_1) + r(S_2).$$

Then $S_1 \cap S_2 \in \mathcal{A}$.

Proof. Let $(E, \mathcal{F})$ be induced by the polymatroid (E, f). Then by Lemma 4.2

$$\beta(S_1 \cap S_2) \leqslant f(S_1 \cap S_2)$$

and by Corollary 4.4 of Korte and Lovász [10]

$$r(S_1) = f(S_1), \qquad r(S_2) = f(S_2)$$

and

$$r(S_1 \cup S_2) = f(S_1 \cup S_2).$$

Hence

$$\beta(S_1 \cap S_2) + r(S_1 \cup S_2) \leqslant f(S_1 \cap S_2) + f(S_1 \cup S_2)$$
$$\leqslant f(S_1) + f(S_2) = r(S_1) + r(S_2).$$

If equality holds, we have

$$\beta(S_1 \cap S_2) = f(S_1 \cap S_2)$$

and hence by Lemma 4.2, $S_1 \cap S_2 \in \mathcal{A}$. $\square$

5. The counterexample

Before describing our construction, we formulate a general construction method for greedoids which will be used.

Lemma 5.1. *Let $(E, \mathcal{F})$ be a greedoid and $A \subseteq E$ a closure feasible set. Let*

$$\mathcal{F}' = \mathcal{F} \cup \{U \subseteq E : \sigma(U) \supseteq A\}.$$

Then $(E, \mathcal{F}')$ is a greedoid.

Proof. First, we observe that if $\sigma(U) \supseteq A$ and $\sigma(V) \supseteq U$ then $\sigma(V) \supseteq A$. We have to show that the greedoid exchange property hold for $\mathscr{F}'$. Let $X, Y \in \mathscr{F}'$ and $|X| < |Y|$. If $\sigma(X) \supseteq A$, then every superset of X is in $\mathscr{F}'$, and we are done. So, suppose that $\sigma(X) \not\supseteq A$. Then $X \in \mathscr{F}$. If $Y \in \mathscr{F}$, we are done again. So, assume that $Y \in \mathscr{F}' - \mathscr{F}$. Then $\sigma(Y) \supseteq A$ and hence $\sigma(X) \not\supseteq Y$ by the above observation. But then for any $x \in Y - \sigma(X)$, we have $X \cup x \in \mathscr{F}$. $\square$

Let us remark that this construction does not in general preserve the interval property, but it will do so in the special application below.

Let $E_0 = \{\bar{p}, \bar{q}, \bar{r}, \bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{c}_1, \bar{c}_2, \bar{c}_3\}$ and let $(E_0, \mathscr{M})$ be the graphic matroid of the graph in Fig. 2.

Let $(E_0, \mathscr{M}')$ be the 6-truncation of $(E_0, \mathscr{M})$. Consider the sets $a_i = \{\bar{a}_i\}$, $b_i = \{\bar{b}_i\}$, $c_i = \{\bar{c}_i\}$, $(i = 1, 2, 3)$ and $x_1 = x_2 = x_3 = \{\bar{p}, \bar{q}, \bar{r}\}$.

Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2, c_3\}$ and $X = \{x_1, x_2, x_3\}$. Let $E = A \cup B \cup C \cup X \subseteq 2^{E_0}$ and let for all $Y \subseteq E$

$$f(Y) = r(\cup Y)$$

where r is the matroid rank function of $(E_0, \mathscr{M}')$. Then f is submodular and monotone on E and hence a polymatroid rank function on E. So (E, f) is a polymatroid.

Let $(E, \mathscr{F})$ be the associated polymatroid greedoid and let $\mathscr{F}'$ be the 6-truncation of

$$\mathscr{F} \cup \{U \subseteq E : \sigma(U) \supseteq A\}.$$

By Lemma 5.1, $(E, \mathscr{F}')$ is a greedoid.

First, we prove some properties of $(E, \mathscr{F})$.

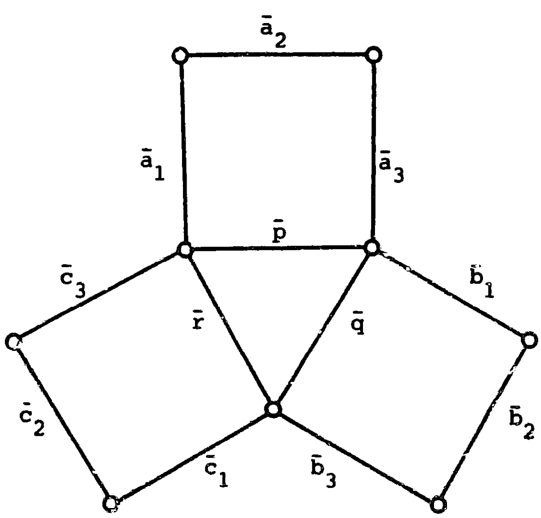


Fig. 2.

Claim 5.1. If $U \in \mathcal{F}$ and $x_i \in U$ for some i, then either $A \subseteq U$ or $B \subseteq U$ or $C \subseteq U$.

Suppose not. We may assume that $U - x_i$ does not contain an x_j. Then $\cup\, U$ contains a unique circuit of the graph, namely $\bar{p}\bar{q}\bar{r}$ and so

$$f(U) = r(\cup\, U) = |\cup\, U| - 1 = |U| + 1 > |U|,$$

a contradiction.

Claim 5.2. If $\sigma(U) \supseteq A$ for some $U \subseteq E$, then either $\sigma(U) = E$ or $A \subseteq U$.

We may assume $U \in \mathcal{F}$ and $|U| \leq 5$. If $U \cap \{X\} = \emptyset$ then $\sigma(U) = E$ or $\sigma(U) = U$ and the assertion is trivial.

Let $x_1 \in U$. Then by Claim 5.1, either $A \subseteq U$ or $B \subseteq U$ or $C \subseteq U$. If $A \subseteq U$ we are done. Assume, say, that $B \subseteq U$. Then $|A \cap U| \leq 1$ as $|U| \leq 5$. But then for any $a_i \in A - U$,

$$\begin{aligned} f(U \cup a_i) &= r(\cup\, U \cup a_i) \\ &= 1 + r(\cup\, U) = 1 + f(U) \end{aligned}$$

and hence $U \cup a_i \in \mathcal{F}$, so $a_i \notin \sigma(U)$, a contradiction.

So we know that

$$\mathcal{F}' = \mathcal{F} \cup \{U \subseteq E : |U| \leq 6,\ U \supseteq A\}.$$

Claim 5.3. $\Gamma(A) = E - A$. This is straightforward.

Claim 5.4. If $A' \subset A$, then $A' \cup \Gamma(A') = A \cup B \cup C$. This is straightforward from Claim 5.1.

Using the previous claims, we prove that $(E, \mathcal{F}')$ has properties (A), (B) and (C).

Claim 5.5. $(E, \mathcal{F}')$ has property (A).

Suppose that Y, $Y - x$, $Y - y \in \mathcal{F}'$, but $Y - x - y \notin \mathcal{F}'$.

Case 1. $Y, Y - x, Y - y \in \mathcal{F}$.

Trivial since $(E, \mathcal{F})$ is a polymatroid greedoid and hence satisfies property (A).

Case 2. $Y \notin \mathcal{F}$.

Then $Y \supseteq A$ by Claim 5.2. Since $Y - x - y \not\supseteq A$, we may assume $Y - x \not\supseteq A$. But then $Y - x \in \mathcal{F}$. So

$$r(Y) \leq |Y| - 1 = |Y - x| = r(Y - x)$$

and so $A \subseteq Y \subseteq \sigma(Y - x)$. By Claim 5.2 again $A \subseteq Y - x$, a contradiction.

Case 3. $Y - x \notin \mathcal{F}$ *but* $Y \in \mathcal{F}$.

Then $Y - x \supseteq A$. So $Y - A \in \mathcal{F}/A$. But $\mathcal{F}/A$ is an interval greedoid in which every singleton is feasible by Claim 5.3. Hence by Lemma 2.2 of Björner et al. [2] $\mathcal{F}/A$ is a matroid and hence $Y - A - x \in \mathcal{F}/A$. But then $Y - x \in \mathcal{F}$, a contradiction.

Claim 5.6. $(E, \mathcal{F}')$ has property (B). Suppose that Y, $Y \cup x$, $Y \cup y$, $Y \cup x \cup y \cup z \in \mathcal{F}'$ but $Y \cup x \cup y \notin \mathcal{F}'$. Then $A \nsubseteq Y \cup x \cup y$ and hence Y, $Y \cup x$, $Y \cup y \in \mathcal{F}$. If $Y \cup x \cup y \cup z \in \mathcal{F}$, then the claim follows, since $(E, \mathcal{F})$ is a polymatroid greedoid and hence has property (B). So, suppose that $Y \cup x \cup y \cup z \notin \mathcal{F}$, and hence $A \subseteq Y \cup x \cup y \cup z$.

Since $A \nsubseteq Y \cup x \cup y$, we have $z \in A$. So $A \nsubseteq Y \cup x$ and so by Claim 5.2 $A \nsubseteq \sigma(Y \cup x)$. Since $y \in \sigma(Y \cup x)$, we have $z \notin \sigma(Y \cup x)$ and hence $Y \cup x \cup z \in \mathcal{F}$. Thus $A \subseteq \sigma(Y \cup x \cup z) = Y \cup x \cup y \cup z$, and so by Claim 5.2, $A \subseteq Y \cup x \cup z$. Similarly $A \subseteq Y \cup y \cup z$ and so $A \subseteq Y \cup z$. Since $Y \cup z \cup x \in \mathcal{F}$, and $\mathcal{F}/A$ is a matroid as above, it follows that $Y \cup z \in \mathcal{F}$.

Let $A' = A - z = A \cap Y$. By property (A), $A' \in \mathcal{F}$.

We show that $Y \subseteq \Gamma(A') \cup A'$. Let $u \in Y - A'$. By Claim 5.3, $u \in \Gamma(A)$ and hence $A \cup u \in \mathcal{F}$. So, $A' \cup u = Y \cap (A \cup u) \in \mathcal{F}$ by property (A), since $Y \cup (A \cup u) = Y \cup A = Y \cup y \in \mathcal{F}$. So $u \in \Gamma(A')$.

Next we show that $x, y \in \Gamma(A')$. We have $A \cup x \in \mathcal{F}$ by Claim 5.3. So $A' \cup x = (Y \cup x) \cap (A \cup x) \in \mathcal{F}$ by property (A), as $(Y \cup x) \cup (A \cup x) = Y \cup y \cup x \in \mathcal{F}$. Hence $x \in \Gamma(A')$ and similarly $y \in \Gamma(A')$.

So $Y \cup x \cup y \subseteq A' \cup \Gamma(A') = A \cup B \cup C$ by Claim 5.4. But $|Y \cup x \cup y| \leqslant 5$ as $Y \cup x \cup y \cup z \in \mathcal{F}'$, and hence by the construction $Y \cup x \cup y \in \mathcal{F}$.

Claim 5.7. $(E, \mathcal{F}')$ has property (C).

Let Y, $Y \cup z$, $Y \cup x \cup y \in \mathcal{F}'$ but assume that $Y \cup x$, $Y \cup x \cup z$ and $Y \cup x \cup y \cup z \notin \mathcal{F}'$. If Y, $Y \cup z$ and $Y \cup x \cup y \in \mathcal{F}$ then we get a contradiction since $(E, \mathcal{F})$ has property (C). If $Y \notin \mathcal{F}$ then $A \subseteq Y$ and hence $A \subseteq Y \cup x$. Since $|Y \cup x| < |Y \cup x \cup y| \leqslant 6$, this implies that $Y \cup x \in \mathcal{F}'$, a contradiction.

We conclude similarly if $Y \cup z \notin \mathcal{F}$ or if $Y \cup x \cup y \notin \mathcal{F}$ and $|Y| \leqslant 3$. So suppose that $|Y| = 4$ and $Y \cup x \cup y \notin \mathcal{F}$. Then $A \subseteq Y \cup x \cup y$ by Claim 5.2

If Y is a basis of $Y \cup x \cup y$ then $\sigma(Y) \supseteq Y \cup x \cup y \supseteq A$ and hence $Y \supseteq A$ by Claim 5.2. So $Y \cup x \in \mathcal{F}'$, a contradiction.

So, suppose Y is not a basis of $Y \cup x \cup y$ and hence $Y \cup y \in \mathcal{F}$ and $\sigma(Y \cup y) \supseteq (Y \cup x \cup y$. Hence by Claim 5.2, $A \subseteq Y \cup y$. Hence $|Y \cap A| \geqslant 2$ and so $B, C \nsubseteq Y$. Since $Y \cup y \cup x \notin \mathcal{F}$, we have

$$f(Y \cup y \cup x) \neq f(Y \cup y) + 1 = 6$$

and hence

$$f(Y \cup y \cup x) \leqslant 5.$$

So

$$f(Y \cup x) \leq 5.$$

Since $Y \cup x \notin \mathcal{F}$, this implies

$$f(Y \cup x) = 4.$$

Also, $Y \cup x \notin \mathcal{F}$ implies that $(Y \cup x) \cap X = \emptyset$. If $X \cap Y \neq \emptyset$ then by Claim 5.1, $A \subseteq Y$ or $B \subseteq Y$ or $C \subseteq Y$. So by the above, $A \subseteq Y$ and hence $Y \cup x \in \mathcal{F}'$, a contradiction.

Assume that $x \in X$. Since $f(Y \cup x) = 4$, by Lemma 4.14 of Korte and Lovász [10] there exists a subset $Y' \subset Y$, $Y' \in \mathcal{F}$, such that $Y' \cup x \in \mathcal{F}$. By Claim 5.1, $A \subseteq Y'$ or $B \subseteq Y'$ or $C \subseteq Y'$ and we conclude as before.

Claim 5.8. $(E, \mathcal{F}')$ is not a trimmed matroid.

We prove this by showing that the inequality of Lemma 4.3 does not hold.

Let $S_1 = B \cup X$, $S_2 = C \cup X$, $S_3 = A \cup X$. Then

$$\beta(S_1 \cap S_2) \geq |S_1 \cap S_2 \cap S_3| = 3,$$
$$r(S_1 \cup S_2) = 6,$$
$$r(S_1) = 4,$$
$$r(S_2) = 4.$$

Thus, we have shown that $(E, \mathcal{F}')$ is not a trimmed matroid, but a local poset greedoid, since properties (A) and (B) hold. Hence, not every local poset greedoid is a trimmed matroid.

Moreover, $(E, \mathcal{F}')$ is not a polymatroid greedoid, but it satisfies also property (C). Hence, polymatroid greedoids cannot be characterized by properties (A), (B) and (C).

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